

09-10-08

Tom Haines Seminar

I.  $F$  is a global field, adeles  $A$ .

$G/F$  connected reductive group, assume  $G$  is semisimple.

$$\text{Let } f = \prod_{v \text{ places}} f_v \in C_c^\infty(G(A))$$

almost  $\prod_v G(F_v)$

$$f_v = f_v^* = 1_{G(\mathcal{O}_v)} \text{ for almost all } v.$$

Consider  $L^2(G) := L^2(G(F) \backslash G(A))$



$R(f)$  for  $f \in L^2(G)$ .

$$(R(f)h)(x) = \int_{G(A)} f(g) h(xg) dg$$

measure on  ~~$G(A)$~~   $G(A)$ .

$$= \int_{G(A)} f(x^{-1}g) h(g) dg$$

$x \mapsto x^{-1}$   $G(A)$

$$= \int_{G(F) \backslash G(A)} \left( \sum_{\gamma \in G(F)} f(x^{-1}\gamma y) \right) \chi(y) dy$$

$y$  measure on  $G(F) \backslash G(A)$

↑  $G(F) \backslash G(A)$

"  $g = \gamma y$ "  $\chi_f(x, y)$

$R(f)$  has kernel  $K_f(x, y)$

We are using:  $\mathcal{J} \subset H$  CG unimodular  $\Rightarrow$

$$\int_{\mathcal{J}/G} f(g) dg = \int_{H/G} \int_{\mathcal{J}/H} f(hg) dh dg$$

Defn:  $f$  is supercuspidal if  $\forall P=MN \subsetneq G, P/F,$

$$\forall x, y \in G(\mathbb{A}), \int f(xny) dn = 0$$

$$N(\mathbb{A}) \xrightarrow{\cong} \prod_v \int_{N(F_v)} f_v(x_v, y_v) dn_v$$

Lemma: ① If  $f_{v_0}$  is supercuspidal, BTW

local analogue

then  $Tf_{v_0}$  is supercuspidal.

②  $f_{v_0}$  = matrix coefficient of s.c. repn  $\overline{\Pi}_{v_0}$

$\Rightarrow f_{v_0}$  is supercuspidal. (Exercise)

Defn:  $h \in L^2(G)$  is cuspidal if  $\forall P=MN \subsetneq G, P/F$

$$\int_{N(F) \backslash N(\mathbb{A})} h(nx) dn = 0 \quad \forall x \in G(\mathbb{A})$$

Let  $L^2_c(F)$  denote the cuspidal functions.

②

Lemma: 1)  $f$  supercuspidal  $\Rightarrow R(f) L^2(G) \subset L^2_{\sigma}(G)$

2)  $R(f) = R_0(f)$  is trace class

$$\text{and } \operatorname{tr} R(f) = \int_{G(F) \backslash G(\mathbb{A})} K_f(x, x) dx$$

$$\text{Also } \operatorname{tr} R_0(f) = \sum_{\pi \in \widehat{G}(\mathbb{A})} \operatorname{tr}(\pi(f))$$

If  $\operatorname{supp}(f)$  meets only  $G(\mathbb{A})$ -conjugacy classes of regular elliptic elements  $\gamma \in G(F)$ , then

$$x \in G(\mathbb{A}) \mapsto \sum_{\gamma \in G(F)_e} [f(x^{-1}\gamma x)]$$

5

$$\int_{G(F) \backslash G(\mathbb{A})} K_f(x, x) dx = \sum_{\{\gamma\} \in G(F)_e / \sim} \int_{G(F) \backslash G(\mathbb{A})} f(x^{-1}\gamma x) dx$$

(3)

So,

Theorem (Deligne - Kazhdan): Let  $f$  s.c., e.g.

$f_{v_1}$  = matrix coeff. of s.c.,

$f_{v_2}$  has support only on elliptic regular conjugacy classes.

Then

we knew

$$\operatorname{tr} R(f) = \sum_{\pi \in L^2_0} m(\pi) \operatorname{tr} \pi(f)$$

$$\pi \in L^2_0$$

$$\sum_{\{\gamma\} \in G(F)_e / \sim} \int_{G_\gamma(F) \backslash G(\mathbb{A})} f(x^{-1}\gamma x) dx$$

$$\text{volume of } G_\gamma(F) \backslash G_\gamma(\mathbb{A})$$

$$\sum_{\{\gamma\} \in G(F)_e / \sim} \tau(\gamma) \langle \phi_\gamma(f) \rangle$$

$$\langle \phi_\gamma(f) \rangle = \int_{G_\gamma(\mathbb{A}) \backslash G(\mathbb{A})} f(x^{-1}\gamma x) dx$$

Next, compare with

(4)

Defn  $\gamma \in G(F)$  is regular elliptic if  $\gamma \notin P(F)$

$\forall P \subset F \quad \gamma \notin G.$

Defn Let  $\gamma \in G(F)$  be semisimple,  $F$  local or global.

$G$  reductive/ $F$ . TFAE:

1)  $C_G(\gamma)$  contains an  $F$ -elliptic maximal torus  $T$

means  $T/\overline{Z(G)}$  is anisotropic.

2)  $\gamma \in$  some elliptic domes

3)  ~~$Z(C_G(\gamma)^\circ)$~~   $Z(C_G(\gamma)^\circ)/\overline{Z(G)}^\circ$  is anisotropic

4)  $\gamma \notin$   $F$ -Levi subgroup  $\forall$  proper  $F$ -Levi  $M$

( $M = C_G(F\text{-split torus})$ )

Aside: let  $S =$  maximal  $F$ -split component of  $\gamma \in C_G(\gamma)^\circ$

let  $M = C(S)$ . Then  $\gamma \in M(F)$

and  $C_M(\gamma)^\circ = C_G(\gamma)^\circ$ . Moreover,  $\gamma$  is elliptic in  $M$ .

Finally,  $\gamma$  is elliptic in  $G \Leftrightarrow M = G$ .

local factoring:  $\tilde{\pi} = \otimes' \tilde{\pi}_v$ ,  $f = \prod_v f_v$

$$\text{tr } \tilde{\pi}(f) = \prod_v \text{tr}(\tilde{\pi}_v(f_v))$$

$$\phi_\chi(f) = \prod_v \phi_{\chi_v}(f_v)$$

Kazhdan Density: Fix  $\chi$ , fix  $f_v$ . Kazhdan density

~~or~~ says: Suppose  $\text{tr}(\tilde{\pi}_v(f_v)) = 0$   $\forall$  irreducible admissible repn  $\tilde{\pi}_v$  of  $G(F_v)$ . Then

$$\phi_{\chi_v}(f_v) = 0 \quad \forall \text{ regular elliptic } \chi_v \in G(F_v).$$

Remark: Converse also holds.

Change notation: ~~F, G~~ local, E, G global.

Choose E, G s.t.  $E_v = F_v$ ,  $\frac{G}{E_v} = G_v$ .

Given  $\gamma = \gamma_v \in G(F_v) = G(E_v)$

Choose  $S = \{v = v_0, v_1, v_2\}$  and  $f_{v_1}$  = matrix coefficient  
of  $\gamma$  s.c.

and  $f_{v_2}$  supported only on regular elliptic elements.

Choose  $\gamma_{v_1}, \gamma_{v_2}$  s.t.  $Q_{v_i}^{\underline{G}(F_{v_i})}(f_{v_i}) \neq 0$

$\underline{G}(F) \hookrightarrow \underline{G}_S^{\text{dense}}$

(7)



09-17-08

## Tom Haines Seminar Part 2

~~Res~~

Thm (DK):  $\prod_v f_v = f \in C_c^\infty(G(A))$  sufficiently nice

$\Rightarrow$

$$\sum_{\substack{\prod \in L^2_0(G) \\ // \\ \gamma \in G(F)_c}} m(\prod) \underbrace{\text{tr } \prod(f)}_{\sim} = \sum_{\substack{\gamma \in G(F)_c \\ //}} \underbrace{\tau(\gamma)}_{\sim} \underbrace{O_\gamma(f)}_{\sim}$$

$$\prod \text{tr } \prod_v (f_v)$$

$$\prod_v O_\gamma(f_v)$$

III.

~~Res~~ Base change Fundamental Lemma (involves stable orbital integrals)

~~Defn~~

Let  $F$  be local field,  $G/F$ .

Assume  $G_{der} = G_{sc}$

Define  $\gamma', \gamma \in G(F)$  are stably conjugate if  $\exists g \in G(F)$

$$\text{s.t. } \gamma' = g \gamma g^{-1}.$$

Fact: Conjugate = Stably Conjugate for  $GL_n$ , for semisimple elements.

At

Non-example :  $SL_2(\mathbb{R})$      $c = \cos(\theta)$

$$s = \sin(\theta)$$

Then  $\begin{pmatrix} c & s \\ -s & c \end{pmatrix} \sim \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$  over  $SL_2(\mathbb{C})$

but not over

$SL_2(\mathbb{R})$ .

Defn Let  $\gamma$  be regular semisimple in  $G(F)$ .

$$SO_\gamma(f) := \sum_{\substack{\gamma' \in G(F)/\sim \\ \gamma' \sim \gamma}} O_{\gamma'}(f) = \int f(x^{-1}\gamma x) dx$$

$(G_\gamma \backslash G)(F)$

Assume  $G$  is quasisplit

Notes:  $G_\gamma(F) \backslash G(F) = G(F) \cdot \gamma$

$$(G_\gamma \backslash G)(F) = \coprod_{\substack{\gamma' \in G(F)/\sim \\ \gamma' \sim \gamma}} G(F) \cdot \gamma'$$

$$\gamma' \sim \gamma$$

$$\gamma' \sim \gamma$$

let  $E/F$  unramified finite, so  $\text{Gal}(E/F) = \langle \theta \rangle$   
is cyclic of order  $r$ .

Let  $G/F$  unramified (i.e. quasisplit), and split over an unramified extension

Define  $N : G(E) \rightarrow G(F)$  by

$$\delta \mapsto \delta \theta(\gamma) \cdots \theta^{r-1}(\gamma) = N(\delta)$$

$$\theta(N\delta) = \delta^* N(\delta) \delta$$

calligraphic  $N$

Fact (Kottwitz):  $N(\delta) \stackrel{\text{st}}{\sim} \text{some } N(\gamma) \in G(F)$ , well-defined  
up to stable conjugacy.

Defn:  $\delta \xrightarrow{\theta\text{-conj}} \delta'$  means  $\delta' = g^{-1}\delta\theta(g)$  and  
From this you get a twisted orbital integral  $T_{\delta'}(\varphi) \sim SO_{\delta'}(\varphi)$

Thus,  $K: \left\{ \begin{array}{l} \text{stable } \theta\text{-conjugacy} \\ \text{classes in } G(E) \end{array} \right\} \xrightarrow{\quad} \left\{ \begin{array}{l} \text{stable conjugacy classes} \\ \text{in } G(F) \end{array} \right\}$

Defn:  $\varphi \in C^\infty(G(E))$ ,  $f \in C^\infty(G(F))$  are

associated if  $\forall$  r.s.  ~~$\delta \in G(F)$~~ ,  $\delta \in G(F)$ ,

$$SO_\delta(f) = \begin{cases} 0 & \text{if } \delta \neq N\delta \\ SO_{N\delta}(\varphi) & \text{if } \delta = N\delta \end{cases}$$

Defn:  $\exists b: \mathcal{H}_K(G(E)) \rightarrow \mathcal{H}_K(G(F))$  on Hecke algebras

where  $K$  is a hyperspecial maximal compact subgroup

Defn:  $\exists b: Z(\mathcal{H}_J(G(E))) \rightarrow Z(\mathcal{H}_J(G(F)))$

where  $J$  is a parahoric

$b$  are base change type maps

Thm (Haines in general)  $\forall \varphi \in Z(\mathcal{H}_J(F(E))),$

$\varphi$  and  $b(\varphi)$  are associated.

Applications :- Shimura Varieties with good reduction at  $p$   
- Base changing Lifting for  $GL_n$ .

#### IV. Stability of $T_e(f)$ (Langlands, Kottwitz)

Let  $F$  global field,  $G/F$ ,  $G$  quasifl. over  $F$ ,  $G = G_{sc}$ .

$$T_e(f) = \sum_{\gamma \in G(F)} \tilde{\tau}(\gamma) O_\gamma(f)$$

Note:  $\gamma, \gamma' \in G(F)$  are  $G(F)$ -conjugate

$\Rightarrow$   $\gamma, \gamma'$  are  $G(A)$  conjugate  $\Rightarrow \gamma, \gamma'$  are

stably conjugate.

$\gamma, \gamma'$   $G(A)$ -conjugate  $\Rightarrow O_\gamma(f) = O_{\gamma'}(f).$

Idea: We're going to group together the  $G(A)$  of elements in  $G(F)$

conjugacy classes in a given stable conjugacy class  
in  $G(F)$ .

Lemma:

① { Conjugacy classes in  $G(F_p)$  which are stably conjugate }  $\xleftarrow{\text{bijective}}$

$$\text{Ker } \{ H^1(F, G) \rightarrow H^1(F_p, G) \}$$

② {conjugacy classes in  $G(F)$  in  $G(A) \cdot \gamma\}$

$$\xleftarrow{\text{bijection}} \ker \left( \ker^{-1}(F, G_\gamma) \rightarrow \ker^{-1}(F, G) \right)$$

$$\text{Therefore, } T_e(f) = \sum_{\gamma \in G(F)} \sum_{\substack{\gamma' \in G(F)/ \\ \sim \\ \gamma'_e \in G(F)_e / \sim^{\text{st}}}} \sum_{\substack{\gamma \sim \gamma' \\ \text{s.t. } \gamma \sim \gamma' \\ \text{G(A)-conj}}} O_{\gamma'}(f)$$

$$= \underbrace{\sum_{\gamma_0} \tilde{\tau}(\gamma_0)}_{b(\gamma_0)} \# \left( \ker^{-1}(F, G_{\gamma_0}) \right) \sum_{\substack{\gamma \in G(F) / \sim \\ \gamma \sim \gamma_0}} O_\gamma(f)$$

$\exists$  obstruction  $\text{obs}(\gamma) \in K(T/F)$